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LETTER TO THE EDITOR

On new transcendents defined by nonlinear ordinary differential equations

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Abstract. The general solution for one of the fourth-order equations is shown to lead to the new transcendent defined by a nonlinear ordinary differential equation.

Fuchs and Poincaré stated the problem of definition of new functions by means of ordinary differential equations (ODEs), necessarily nonlinear [1]. These ODEs have to have two important properties: irreducibility of them and uniformization of their solutions. As to the first property, it means that there exists no transformation, again within a precise class, reducing any of these equations either to a linear equation or to another order equation. Consequently, the general solutions of these equations have no explicit expressions, they are just defined by the equations themselves [1, 2]. The second property corresponds to the Painlevé property of an ODE because this property leads to the absence of movable critical singularities in its general solution.

The second-order ODE class was studied by Painlevé and his school. They began a study of these ODEs almost a century ago. It is likely they had two related objectives: to classify the second-order equations (of a certain form) on the basis of their possible singularities, and to identity equations of second order that define essentially new functions. Their work seems to stand as one of the noteworthy successes of 20th century mathematics. They showed that (up to transformations) there are exactly six second-order equations that define new functions. The functions defined by them are now called the six Painlevé transcendents. Later, Bureau extended Painlevé's first objective, and gave a partial classification of third-order equations [3–5]. The results of Painlevé and his collaborators led to the problem of finding other new functions that could be defined by nonlinear ODEs like the Painlevé transcendents. However, despite huge efforts, no new function has yet been found. In fact, no irreducible equation has been discovered since 1906 [1].

Although the Painlevé equations were first discovered from strictly mathematical considerations, they have recently appeared in several physical applications [6].

In this letter we are going to show that the general solution for one fourth order equation introduced in [7] gives a new function defined by nonlinear ODEs like the Painlevé transcendents.

In recent work [7] a hierarchy was introduced that takes the form

$$d^{n+1}(u) - \frac{z}{2} = 0 \qquad (n = 1, 2, \ldots) \tag{1}$$

where the operator d^n is determined by the formula

$$\frac{\mathrm{d}}{\mathrm{d}z}d^{n+1}(u) = d^n_{zzz} + 4ud^n_z + 2u_z d^n$$
$$d^0 = \frac{1}{2} \qquad d^1 = u. \tag{2}$$

We have the first Painlevé equation

$$u_{zz} + 3u^2 - \frac{z}{2} = 0 \tag{3}$$

from equations (1) at $n = 1$.

If we take $n = 2$ in equations (1) we obtain the fourth order equation in the form

$$u_{zzzz} + 5u_z^2 + 10uu_{zz} + 10u^3 - \frac{z}{2} = 0. \tag{4}$$

It is well known that equation (3) determines a new function which is the Painlevé transcendent. However, there is a question about what functions are determined by equation (1) at $n \geqslant 2$.

To answer this question it is necessary first to check the Painlevé property for equations (1) and second to show that the general solutions of these equations are the essentially transcendental functions of their initial conditions [8].

Recently [9], we have studied some properties of equations (1) and now we can affirm that these equations possess the Painlevé property because of the following reasons.

First, these equations were obtained as reductions of nonlinear partial differential equations which were solved by inverse scattering transform. Following the famous conjecture of Ablowitz *et al* [10, 11] we can expect that equations (1) have to have the Painlevé property.

Second, using the algorithm of Conte *et al* [12] we checked equation (4) by the Painlevé test which is the necessary condition for integrability of this equation. It turned out that equation (4) passes the Painlevé test.

We also found the Lax pairs for equations (1), which give the possibility to solve these equations. It is known [1] that a 'good' Lax pair is the sufficiency condition for the integrability of the equation. As this takes place the application of the Gelfand–Levitan–Marchenko integral equation gives the algorithm for solving the Caushy problem and strict proof of the Painlevé property for nonlinear equations. We obtained the Lax pairs that have the same form for all equations of hierarchy (1). These Lax pairs are 'good' because one of them was used for the solution of equation (3).

Taking into account the above-mentioned reasons, one can see that the Painlevé property (or property of uniformization) for the solutions of hierarchy (1) is confirmed.

Now the solution of the problem of finding new functions determined by nonlinear ordinary differential equations (1) reduces to the investigation of the functional dependence upon the general solutions from the initial conditions.

Three different cases are possible [8]. In the first case, the general solution of the equation can have the rational or algebraic function of arbitrary constants. This case cannot lead to the new function. In the second case, the general solution does not have any rational dependence on initial conditions but the equation can have some integral. In this case some arbitrary constants can enter in algebraic form of the integral. This case leads to the semi-transcendental function of the general solution and also does not give any new function. In the third case, the dependence of the general solution of the equation on the initial conditions is such that it differs from the first and second cases. Concerning solutions of this case, one can say, this one is the essentially transcendental function.

Let us consider dependent solutions of (1) on their initial conditions, taking into account the above-listed cases. We have the following theorem.

Theorem. The general solutions of equations (1) at $n = 1$ and $n = 2$ are the essentially transcendental functions with respect to constants of integration.

Proof. We are going to prove this theorem at $n = 1$ and $n = 2$ because the cases at other n are proved by this analogy.

Equation (3) was studied by Painlevé who found that the general solution of this equation gives a new function. He used the variables [8]

$$u = \mu^{-2}u' \qquad z = \mu z' \tag{5}$$

where μ is some parameter, and transformed equation (4) into the following equation

$$u_{zz} + 3u^2 - \frac{\mu^5}{2}z = 0 \tag{6}$$

(the primes of the variables are omitted). Assuming $\mu = 0$ in equation (6), one can see this equation has a general solution in the form of an elliptic Jacobi function. This solution has no rational dependence on the initial conditions. Therefore, the general solution of (6) at $\mu \neq 0$ also has no rational dependence on arbitrary constants [8].

In a similar manner, (1) can be presented in the form

$$d^{n+1}(u) - \frac{\mu^{(2n+3)}z}{2} = 0 \tag{7}$$

if we use variables (5).

It is easy to see that (7) are transformed to the stationary KdV hierarchy at $\mu = 0$. As an example, we have the equation

$$u_{zzzz} + 5u_z^2 + 10uu_{zz} + 10u^3 - \frac{\mu^7}{2}z = 0 \tag{8}$$

from (7) at $n = 2$ which corresponds to the stationary KdV equation of fifth order

$$u_{zzzz} + 5u_z^2 + 10uu_{zz} + 10u^3 = 0 \tag{9}$$

in the case $\mu = 0$. The solution of (9) was studied in detail by Dubrovin [13]. He found that the solution of (9) can be expressed by the theta function on the Riemann surface [13–15]:

$$u = \frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\theta(az + z_0) \tag{10}$$

where $\theta(z)$ is the theta function on the Riemann surface, a is the vector of periods of some normalized differential, z_0 is the arbitrary two-dimensional vector [13]. Solution (10) of equation (9) is the transcendental function with respect to their arbitrary constants [13] and consequently the solution of equation (8) has no rational dependence on the arbitrary constants at $\mu \neq 0$ too. We obtain that the general solution of equation (4) does not correspond to the first case.

Let us show that equation (1) does not have any integral at $n = 1$. In fact, this theorem has been proved [8, 16] but we want to use another approach here so that one can apply this approach to the proof at other values n for equations (1). Let us assume that (1) at $n = 1$ has an integral in the form

$$P_1(u, u_z, z) = C_1 \tag{11}$$

where C_1 is an arbitrary constant and P_1 is some polynomial of u, u_z and z. Equation (11) leads to the following equation

$$E_1 = \frac{\partial P_1}{\partial z} + \frac{\partial P_1}{\partial u} u_z + \frac{\partial P_1}{\partial u_z} u_{zz} = 0 \tag{12}$$

which corresponds to equation (3) so that we have the equality

$$E_1 = Q_1 \left(u_{zz} + 3u^2 - \frac{z}{2} \right) \tag{13}$$

where Q_1 can depend on u, u_z and z too.

One can find the equality

$$\frac{\partial P_1}{\partial u_z} = Q_1 \tag{14}$$

from equation (13), so therefore equation (13) can be written in the form

$$\frac{\partial P_1}{\partial z} + \frac{\partial P_1}{\partial u} u_z - \left(3u^2 - \frac{z}{2}\right) \frac{\partial P_1}{\partial u_z} = 0 \tag{15}$$

Let us look for the solution of equation (15) in the form [8, 16].

$$P_1 = u_z^m + q_1(u, z) m_z^{m-1} + \cdots + q_{m-1} u_z + q_m(u, z). \tag{16}$$

Substituting (16) into (15) and equating the expressions of the same powers of u_z to zero gives the following set of equations:

$$\frac{\partial q_1}{\partial u} = 0 \tag{17}$$

$$\frac{\partial q_2}{\partial u} + \frac{\partial q_1}{\partial z} - 3mu^2 + \frac{1}{2} mz = 0 \tag{18}$$

$$\frac{\partial q_3}{\partial u} + \frac{\partial q_2}{\partial z} - 3(m-1)u^2 q_1 + \frac{1}{2}(m-1) z q_1 = 0 \tag{19}$$

$$\frac{\partial q_{k+1}}{\partial u} + \frac{\partial q_k}{\partial z} - 3(m-k+1)u^2 q_{k-1} + \frac{1}{2}(m-k+1) z q_{k+1} = 0 \qquad (k = 3, \ldots, m-1) \tag{20}$$

$$\frac{\partial q_m}{\partial z} - 3u^2 q_{m-1} + \frac{1}{2} z q_{m-1} = 0. \tag{21}$$

These equations can be solved sequentially except for equation (21). We have

$$q_1 = f_1(z) \tag{22}$$

from equation (17). Substituting (22) into (18) gives the solution

$$q_2 = mu^3 - \left(\frac{\mathrm{d} f_1}{\mathrm{d} z} + \frac{1}{2} zm \right) u + f_2(z). \tag{23}$$

We also obtain from equations (19) and (20)

$$q_3 = (m-1) f_1 u^3 + \frac{1}{2} \left(\frac{\mathrm{d}^2 f_1}{\mathrm{d} z^2} + \frac{m}{2} \right) u^2 - \left(\frac{\mathrm{d} f_2}{\mathrm{d} z} + \frac{1}{2}(m-1) f_1 z \right) u + f_3(z) \tag{24}$$

$$\begin{aligned} q_4 = {} & \frac{1}{2} m(m-2) u^6 - \frac{1}{2} m(m-2) z u^4 - \frac{3}{4}(m-2) u^4 \frac{\mathrm{d} f_1}{\mathrm{d} z} + (m-2) f_2 u^3 \\ & + \frac{1}{4}(m-2) z u^2 \frac{\mathrm{d} f_1}{\mathrm{d} z} + \frac{1}{8} m(m-2) z^2 u^2 \\ & - \frac{1}{6} u^3 \frac{\mathrm{d}^3 f_1}{\mathrm{d} z^3} + \frac{1}{2} u^2 \frac{\mathrm{d}^2 f_1}{\mathrm{d} z^2} - u \frac{\mathrm{d} f_3}{\mathrm{d} z} + f_4 \end{aligned} \tag{25}$$

and

$$q_5 = \frac{1}{2}(m-1)(m-3) f_1 u^6 + \frac{1}{4}\left(m - \frac{13}{5}\right) m u^5 + \frac{9}{20}\left(m - \frac{8}{3}\right) u^5 \frac{\mathrm{d}^2 f_1}{\mathrm{d}z^2} + \frac{1}{2}(m-1)(m-3) f_1 z u^4 - \frac{3}{4}(m-3) u^4 \frac{\mathrm{d} f_2}{\mathrm{d}z} + \cdots - \frac{1}{2}(m-3) f_3 z u + \frac{1}{24} u^4 \frac{\mathrm{d}^4 f_1}{\mathrm{d}z^4} - \frac{1}{6} u^3 \frac{\mathrm{d}^3 f_2}{\mathrm{d}z^3} + \frac{1}{2} u^2 \frac{\mathrm{d}^2 f_3}{\mathrm{d}z^2} - u \frac{\mathrm{d} f_4}{\mathrm{d}z} + f_5. \tag{26}$$

Solutions (25) and (26) give the form of other solutions of equations (20). In fact, solutions of these equations can be found for $k = 0, \ldots, m$. However, as this takes place the solutions q_m and q_{m-1} have to satisfy equation (21) if P_1 is an integral of equation (4). One can see this equation is not satisfied.

Let us show this. First, let $f_1 \neq 0$, then using the method of mathematical induction we have

$$q_{2k} = A_{3k} u^{3k} + \cdots \qquad q_{2k+1} = B_{3k} f_1 u^{3k} + \cdots \tag{27}$$

where A_{3k} and B_{3k} are constants. Substituting (27) into (21) gives a contradiction.

Then let $f_1 = 0$ so taking into account the method of mathematical induction again we obtain

$$q_{2k} = A_{3k} u^{3k} - A_{3k-2} z u^{3k-2} + A_{3k-3} f_2 u^{3k-3} + A_{3k-4} z^2 u^{3k-4} + \cdots \tag{28}$$

and

$$q_{2k+1} = B_{3k-1} u^{3k-1} - B_{3k-2} \frac{\mathrm{d} f_2}{\mathrm{d}z} u^{3k-2} + \cdots \tag{29}$$

where A_{3k}, B_{3k-1} and so on are constants of the same sign. Substituting q_m and q_{m-1} into (21) leads to a contradiction in this case too. These contradictions prove the theorem.

Let us note that the integral of equation (6) at $\mu = 0$ can be found. This can be presented in the form

$$P_1 = u_z^2 + 2u^3 = C_1. \tag{30}$$

Now let us consider equation (1) at $n = 2$. We assume that there is some integral

$$P_2(u, u_z, u_{zz}, u_{zzz}, z) = C_2 \tag{31}$$

of (4), where C_2 is an arbitrary constant and P_2 is some polynomial of u, u_z, u_{zz}, u_{zzz} and z. Equation (31) leads to the following equation

$$E_2 = \frac{\partial P_2}{\partial z} + \frac{\partial P_2}{\partial u} u_z + \frac{\partial P_2}{\partial u_z} u_{zz} + \frac{\partial P_2}{\partial u_{zz}} u_{zzz} + \frac{\partial P_2}{\partial u_{zzz}} u_{zzzz} = 0 \tag{32}$$

so that we have the equality

$$E_2 = Q_2 \left(u_{zzzz} + 5u_z^2 + 10 u u_{zz} + 10u^3 - \frac{z}{2} \right) \tag{33}$$

where Q_2 can depend on u, u_z, u_{zz}, u_{zzz} and z . One can find the equality

$$\frac{\partial P_2}{\partial u_{zzz}} = Q_2 \tag{34}$$

from equation (33). Thus equation (33) can be written in the form

$$\frac{\partial P_2}{\partial z} + \frac{\partial P_2}{\partial u} u_z + \frac{\partial P_2}{\partial u_z} u_{zz} + \frac{\partial P_2}{\partial u_{zz}} u_{zzz} - \left(5u_z^2 + 10 u u_{zz} + 10u^3 - \frac{z}{2} \right) \frac{\partial P_2}{\partial u_{zzz}} = 0. \tag{35}$$

Let us look for the solution P_2 of equation (35) in the form

$$P_2 = r_0 u_{zzz}^m + r_1 u_{zzz}^{m-1} + \cdots + r_{m-1} u_{zzz} + r_m \tag{36}$$

where

$$r_k = r_k(u, u_z, u_{zz}, z). \tag{37}$$

Substituting (36) into (35) and equating of the same powers u_{zzz} to zero gives the following set of equations:

$$\frac{\partial r_0}{\partial u_{zz}} = 0 \tag{38}$$

$$\frac{\partial r_0}{\partial z} + \frac{\partial r_0}{\partial u} u_z + \frac{\partial r_0}{\partial u_z} u_{zz} + \frac{\partial r_1}{\partial u_{zz}} = 0 \tag{39}$$

$$\frac{\partial r_2}{\partial u_{zz}} + \frac{\partial r_1}{\partial u_z} u_{zz} + \frac{\partial r_1}{\partial u} u_z + \frac{\partial r_1}{\partial z} = m r_0 \left(5u_z^2 + 10uu_{zz} + 10u^3 - \frac{z}{2}\right) \tag{40}$$

$$\frac{\partial r_3}{\partial u_{zz}} + \frac{\partial r_2}{\partial u_z} u_{zz} + \frac{\partial r_2}{\partial u} u_z + \frac{\partial r_2}{\partial z} = (m-1) r_1 \left(5u_z^2 + 10uu_{zz} + 10u^3 - \frac{z}{2}\right) \tag{41}$$

$$\frac{\partial r_{k+1}}{\partial u_{zz}} + \frac{\partial r_k}{\partial u_z} u_{zz} + \frac{\partial r_k}{\partial u} u_z + \frac{\partial r_k}{\partial z} - (m-k+1) r_{k-1} \times \left(5u_z^2 + 10uu_{zz} + 10u^3 - \frac{z}{2}\right) \qquad (k = 3, \ldots, m-1) \tag{42}$$

$$\frac{\partial r_m}{\partial u_z} u_{zz} + \frac{\partial r_m}{\partial u} u_z + \frac{\partial r_m}{\partial z} = r_{m-1} \left(5u_z^2 + 10uu_{zz} + 10u^3 - \frac{z}{2}\right). \tag{43}$$

These equations can be solved sequentially except for equation (43).

Solution of equation (38) can be presented in the form

$$r_0 = r_0(u, u_z, z) \tag{44}$$

One can see that

$$r_1 = -\frac{1}{2} \frac{\partial r_0}{\partial u_z} u_{zz}^2 - b_0 u_{zz} + f_1(u, u_z, z) \tag{45}$$

from equation (39), where

$$b_0 = \frac{\partial r_0}{\partial z} + \frac{\partial r_0}{\partial u} u_z.$$

Using solution (45) one can obtain

$$r_m = (-1)^m \frac{\partial^m r_0}{\partial u_z^m} u_{zz}^{2m} + \cdots. \tag{46}$$

Substituting (46) into (43) leads to the only power $2m+1$ of u_{zz}. Therefore,

$$r_0 = a_0(u, z) u_z^m + a_1(u, z) u_z^m + \cdots + a_m(u, z). \tag{47}$$

However, as this takes place we have from equation (43)

$$\frac{\mathrm{d}^m b_0}{\mathrm{d} u_z^m} = 0. \tag{48}$$

In fact one can find the general form of dependence r_0 upon u, u_z and z in the form of polynomial taking into account equations (39)–(42) and (48) but one can note that r_0 is contained in r_k as linear expression and without loss of generality let us take

$$r_0 = u_z^m. \tag{49}$$

Now one can obtain that

$$r_1 = -\tfrac{1}{2} m u_z^{m-1} u_{zz}^2 + f_1(u, u_z, z). \tag{50}$$

Taking into account equation (40) we can obtain

$$r_2 = \frac{1}{8} m(m-1) u_z^{m-2} u_{zz}^4 + \frac{1}{2}(10 m u u_z^m - \frac{\partial f_1}{\partial u_z}) u_{zz}^2$$
$$+ u_{zz}\left[m u_z^m \left(5u_z^2 + 10u^3 - \frac{z}{2}\right) - \left(\frac{\partial f_1}{\partial u} u_z + \frac{\partial f_1}{\partial z}\right)\right] + f_2(u, u_z, z). \tag{51}$$

One can assume that the solution for r_k takes the form

$$r_k = a_k u_{zz}^{2k} + b_k u_{zz}^{2k-2} + c_k u_{zz}^{2k-3} + \cdots \tag{52}$$

and one can find the recursion relations from equation (42) for a_{k+1}, b_{k+1} and c_{k+1} in the form

$$a_{k+1} = -\frac{1}{(2k+2)} \frac{\partial a_k}{\partial u_z} \tag{53}$$

$$b_{k+1} = \frac{1}{2k}\left[10(m-k+1) a_{k-1} u - \frac{\partial b_k}{\partial u_z}\right] \tag{54}$$

$$c_{k+1} = \frac{1}{(2k-1)}\left[(m-k+1) a_{k-1}\left(5u_z^2 + 10u^3 - \frac{z}{2}\right) - \frac{\partial c_k}{\partial u_z} - \left(\frac{\partial b_k}{\partial z} + u_z \frac{\partial b_k}{\partial u}\right)\right]. \tag{55}$$

Assuming $k = m-1$ one can obtain the formulae for a_m, b_m and c_m from equations (53)–(55).

Solutions r_m and r_{m-1}, on the other hand, have to satisfy equation (43). Taking this into account, we have

$$\frac{\partial a_m}{\partial u_z} = 0 \tag{56}$$

$$\frac{\partial b_m}{\partial u_z} = 10 u a_{m-1} \tag{57}$$

$$\frac{\partial c_m}{\partial u_z} + \frac{\partial b_m}{\partial z} + u_z \frac{\partial b_m}{\partial u} = a_{m-1}\left(5u_z^2 + 10u^3 - \frac{z}{2}\right). \tag{58}$$

One can see that the coefficients a_k are determined by the formula

$$a_k = (-1)^k \frac{m(m-1)\ldots(m-k+1)}{2^k k!} u_z^{m-k} \tag{59}$$

so that we have

$$a_{m-1} = (-1)^{m-1} \frac{m}{2^{m-1}} u_z \qquad a_m = (-1)^m \frac{1}{2^m}. \tag{60}$$

One can also find that

$$b_m = (-1)^{m-1} \frac{m}{2^{m-1}} [5 u u_z^2 + g(u, z)] \tag{61}$$

from equation (57). Substituting (61) into (58) gives

$$g(u, z) = \tfrac{5}{2} u^4 - \tfrac{1}{2} z u + p(z) \tag{62}$$

where $p(z)$ is a function of integration over u.

Using the method of mathematical induction one can obtain from equation (54) the coefficients b_k

$$b_k = (-1)^{k-1} \frac{m(m-1)\ldots(m-k+1)}{2^{k-1}(k-1)!} u_z^{m-k}\left[5 u u_z^2 + \frac{5}{2} u^4 - \frac{1}{2} z u + p(z)\right]. \tag{63}$$

We also have

$$b_1 = mu_z^{m-1}(5uu_z^2 + \tfrac{5}{2}u^4 - \tfrac{1}{2}zu + p(z)) \tag{64}$$

from equation (63). Taking into account equation (64) we obtain

$$c_2 = -mu_z^{m-1}\left(\frac{\partial p}{\partial z} - \frac{1}{2}u\right) \tag{65}$$

from (55).

Assuming that

$$c_k = (-1)^{k-1}A_k u_z^{m-k+1}\left(\frac{\partial p}{\partial z} - \frac{1}{2}u\right) \tag{66}$$

(where A_k is some constant) leads by the method of mathematical induction to the relation

$$c_{k+1} = (-1)^k A_{k+1} u_z^{m-k}\left(\frac{\partial p}{\partial z} - \frac{1}{2}u\right) \tag{67}$$

where

$$A_{k+1} = \frac{(m-k+1)}{2k-1}\left[A_k + \frac{m(m-1)(m-k)}{2^{k-1}(k-1)!}\right]. \tag{68}$$

One can obtain from equation (67)

$$c_m = (-1)^{m-1}A_m u_z\left(\frac{\partial p}{\partial z} - \frac{1}{2}u\right). \tag{69}$$

Substituting (60), (61) and (69) into (58) leads to a contradiction because we have

$$\left(A_m + \frac{m}{2^{m-1}}\right)\left(\frac{\partial p}{\partial z} - \frac{1}{2}u\right) \neq 0. \tag{70}$$

This contradiction shows that the integral of equation (4) in the form (31) does not exist. This proves the theorem at $n = 2$. □

Let us note that equation (8) at $\mu = 0$ has an integral in the form

$$P_2 = u_z u_{zzz} - \tfrac{1}{2}u_{zz}^2 + 5uu_z^2 + \tfrac{5}{2}u^4 = C_2. \tag{71}$$

Thus we have found that the general solution of (4) does not belong to the second class as a function of the initial conditions. This indicates that the general solution of (4) corresponds to the third case.

Thus we have shown that the general solution of (4) is the essentially transcendental function with respect to the constants of integration.

In the approximate limit, the general solution of this equation can be presented via the theta function of the Riemann surface which is the semi-transcendental function with respect to the constants of integration. Consequently, this solution has got no rational dependence on the constants.

We have shown that equation (4) has no first integral. Consequently, the general solution of this equation is the essentially transcendental function with respect to the constants of integration. This solution belongs to the class of functions like the six Painlevé transcendents.

We believe that every general solution of equations (1) is the transcendents defined by nonlinear ordinary differential equations too and therefore we have to have an infinite number of such transcendents. We suppose that these transcendents are new because the six Painlevé transcendents are essentially transcendental functions of two constants but the general solutions of equation (4) are transcendents of four constants.

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